Energy dissipation in body-forced plane shear flow

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We study the problem of body-force driven shear flows in a plane channel of width ℓ with free-slip boundaries. A mini-max variational problem for upper bounds on the bulk time averaged energy dissipation rate ϵ is derived from the incompressible Navier-Stokes equations with no secondary assumptions. This produces rigorous limits on the power consumption that are valid for laminar or turbulent solutions. The mini-max problem is solved *exactly* at high Reynolds numbers $Re = U\ell/\nu$, where U is the rms velocity and ν is the kinematic viscosity, yielding an explicit bound on the dimensionless asymptotic dissipation factor $\beta = \epsilon\ell/U^3$ that depends only on the "shape" of the shearing body force. For a simple half-cosine force profile, for example, the high Reynolds number bound is $\beta \leq \pi^2/\sqrt{216} = .6715\ldots$. We also report extensive direct numerical simulations for this particular force shape up to $Re \approx 400$; the observed dissipation rates are about a factor of three below the rigorous high-Re bound. Interestingly, the high-Re optimal solution of the variational problem bears some qualitative resemblence to the observed mean flow profiles in the simulations. These results extend and refine the recent analysis for body-forced turbulence in J. Fluid Mech. **467**, 289-306 (2002).

1. Introduction

Bounds on the energy dissipation rate for statistically stationary flows belong to the small class of rigorous results for turbulence that can be derived directly from the incompressible Navier-Stokes equations without introducing any supplementary hypotheses or uncontrolled approximations. Quantitative approaches are mostly based on variational formulations as have been used in a variety of boundary-driven turbulent flows; see, e.g., Howard (1972), Busse (1978), Doering & Constantin (1994), Nicodemus *et al* (1998), Kerswell (1998). More recently Childress, Kerswell & Gilbert (2001) and Doering & Foias (2002) extended these analyses to body-forced flows in a fully periodic domain. The motivation for such studies is to consider mathematically well-defined and tractable models for (almost) homogeneous and (almost) locally isotropic stationary turbulence when boundaries are far away.

Define the Reynolds number $Re = U\ell/\nu$, where U is the steady state rms velocity, ℓ is the longest characteristic length scale in the body-force function and ν is the kinematic viscosity. For the three-dimensional Navier-Stokes equations, Doering & Foias (2002) found that the energy dissipation rate per unit mass ϵ satisfies

$$\epsilon \leq c_1\nu\frac{U^2}{\ell^2} + c_2\frac{U^3}{\ell} \tag{1.1}$$

where the coefficients c_1 and c_2 depend only on the functional shape of the body-force, and not on any other parameters or on any ratios involving the (say, rms) amplitude F of the force or the overall system size L—which could be arbitrarily larger than ℓ. (We give a more precise definition of the "shape" of the forcing function below, or else see Doering & Foias (2002).) In terms of the dimensionless dissipation ratio $\beta = \epsilon\ell/U^3$, this result is $\beta \leq c_1/Re + c_2$, an estimate in qualitative accord with theoretical, computational, and experimental result for homogeneous isoptropic turbulence (Frisch (1995), Sreenivasan (1984), Sreenivasan (1998)). The analysis in Childress, Kerswell & Gilbert (2001) focuses on dissipation estimates in terms of the true control parameter for such systems, the Grashof number $Gr = F\ell^3/\nu^2$, but those results are less easily interpreted in terms of conventional ideas for homogeneous isotropic turbulence.

In this paper we refine and develop the approach in Doering & Foias (2002) for the particular example of flow between free-slip boundary planes driven by a steady volume forcing density. Homogeneous shear turbulence is of interest in its own right, and by setting a uniform direction of the force we simplify some of the analysis allowing us to improve the bounds and make quantitative comparison with direct numerical simulations. The analysis produces rigorous limits that are approached within about a factor of three even at the moderate values of Re (up to 400) that we are able to reach computationally (in these runs the usual Taylor microscale Reynolds number $R_\lambda \approx 100$).

The rest of this paper is outlined as follows. In the next section we describe the model in detail and introduce some notation and definitions. In section 3 we derive the mini-max problem for the dissipation rate bounds. Elementary analysis quickly produces estimates which are then refined to the optimal bound (within this variational formulation) in the limit of $Re \to \infty$. The final section 4 is a comparison of the results with the computational data and a brief discussion of the results.

2. Preliminaries

The three-dimensional dynamics of the flow is governed by the Navier-Stokes equations for an incompressible Newtonian fluid,

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \nu\nabla^2\mathbf{u} + \mathbf{f}, \qquad \nabla\cdot\mathbf{u} = 0, \tag{2.1}$$

where the velocity field is $\mathbf{u}(\mathbf{x},t) = \mathbf{e}_x u_x + \mathbf{e}_y u_y + \mathbf{e}_z u_z$, the pressure field is $p(\mathbf{x},t)$, and the kinematic viscosity is ν. The flow is in the slab, $[0,L_x]\times[0,\ell]\times[0,L_z]$ with periodic boundary conditions in the x (streamwise) and z (spanwise) directions. On the bottom and top surfaces at $y=0$ and $y=\ell$, we take the free-slip boundary conditions $u_y = 0$ and $\partial_y u_x = 0 = \partial_y u_z$.

The steady body-force shearing the fluid is taken to be of the form

$$\mathbf{f}(\mathbf{x}) = F\phi\left(\frac{y}{\ell}\right)\mathbf{e}_x\,. \tag{2.2}$$

The length scale ℓ is the longest length scale in the forcing function. The dimensionless *shape* function $\phi : [0,1] \to \mathcal{R}$ satisfies homogeneous Neumann boundary conditions with zero mean:

$$\phi'(0) = 0 = \phi'(1), \qquad \int_0^1 \phi(\eta)d\eta = 0. \tag{2.3}$$

Technically we require that ϕ is a square integrable function, i.e., $\phi \in L^2[0,1]$, but in practice we are interested in even smoother functions whose Fourier transforms are effectively supported on low wavenumbers. The amplitude F is specified uniquely for a

given $\mathbf{f}$ when we fix the normalization of the shape function by

$$1 = \int_0^1 \phi(\eta)^2 d\eta \,. \tag{2.4}$$

We also introduce the dimensionless "potential" for the body-force shape function via

$$\mathbf{f}(\mathbf{x}) = \nabla \times \left[-F\Phi\left(\frac{y}{\ell}\right) \mathbf{e}_z \right] \,. \tag{2.5}$$

Then $\Phi \in H^1[0,1]$, the space of functions with square integrable first derivatives, so $\phi = -\Phi'$ and, without loss of generality because ϕ has zero mean, it satisfies homogeneous Dirichlet boundary conditions, $\Phi(0) = 0 = \Phi(1)$.

At sufficiently high forcing amplitude, a finite perturbation causes transition to turbulence and the imposed driving sustains the turbulent state assuring statistical stationarity of the turbulent flow. In the following $\langle \cdot \rangle$ denotes the space-time average. Using the root mean square value $U = \sqrt{\langle \mathbf{u}^2 \rangle}$ of the total velocity field—including both a possible mean flow and turbulent fluctuations—and the length scale ℓ in the force, the Reynolds number is

$$Re = \frac{U\ell}{\nu} \,. \tag{2.6}$$

The energy dissipation per unit mass is $\epsilon = \nu \langle |\nabla \mathbf{u}|^2 \rangle$ and we define the dimensionless dissipation factor β via

$$\epsilon = \beta \, \frac{U^3}{\ell} \,. \tag{2.7}$$

Our aim is to derive bounds on β as a function of Re and as a functional of the shape ϕ of the driving force.

3. Bounds for the energy dissipation

The calculation of upper bounds on β proceeds in two steps. First is the derivation of a variational expression and second is the determination of rigorous estimates for it.

3.1. *The variational problem*

From the averaged power balance in the Navier-Stokes equations, the energy dissipation rate per unit mass ϵ is†

$$\epsilon = \nu \left\langle |\nabla \mathbf{u}|^2 \right\rangle = F \left\langle \phi u_x \right\rangle . \tag{3.1}$$

Another expression for the forcing amplitude F can be obtained by projecting onto the momentum equation. Specifically, we project the streamwise component of the Navier-Stokes equations onto a mean zero multiplier function $\psi \in H^2[0,1]$ (a function whose second derivative is square integrable) satisfying homogeneous Neumann boundary conditions $\psi'(0) = 0 = \psi'(1)$. The multiplier function ψ must *not* be orthogonal to the shape function ϕ; we consider only $\langle \phi\psi \rangle \neq 0$. It is also convenient to introduce the derivative of the multiplier function, $\Psi = \psi' \in H^1[0,1]$, satisfying homogeneous Dirichlet boundary conditions $\Psi(0) = 0 = \Psi(1)$. The inner product of ϕ and ψ is the inner product of Φ and Ψ, i.e., $\langle \Phi\Psi \rangle = \langle \phi\psi \rangle \neq 0$.

† Strictly speaking we are also assuming that the long time averages exist and that this relation is an equality for the solutions, rather than just an inequality. That is, for weak solutions of the Navier-Stokes equations it is only known that $\epsilon \leq F \langle \phi u_x \rangle$. These mathematical technicalities do not alter the ultimate bounds that we will derive in this paper.

Take the inner product of the Navier-Stokes equation (2.1) with $\psi(y/\ell)\mathbf{e}_x$, integrate over the volume utilizing appropriate integrations by parts, and take the long time average to obtain the relation

$$-\left\langle \frac{1}{\ell}\psi' u_x u_y \right\rangle = \left\langle \frac{\nu}{\ell^2}\psi'' u_x \right\rangle + F\left\langle \phi\psi\right\rangle . \tag{3.2}$$

This may be solved for the strength of the applied force F which when inserted into (3.1) yields

$$\epsilon = -\frac{\langle \phi u_x\rangle \left\langle \frac{1}{\ell}\psi' u_x u_y + \frac{\nu}{\ell^2}\psi'' u_x\right\rangle}{\langle \phi\psi\rangle}. \tag{3.3}$$

While the force amplitude F is not explicitly displayed in (3.3) anymore, it is implicitely present through the constraint that the root mean square value of the velocity field is U. Dividing by U^3/ℓ, we produce an expression for the dimensionless dissipation factor β,

$$\beta = \frac{\epsilon\ell}{U^3} = -\frac{\left\langle \phi(\frac{u_x}{U})\right\rangle \left\langle \psi'(\frac{u_x}{U})(\frac{u_y}{U}) + \frac{1}{Re}\psi''(\frac{u_x}{U})\right\rangle}{\langle \phi\psi\rangle}. \tag{3.4}$$

Changing now to normalized velocities $u\mathbf{e}_x + v\mathbf{e}_y + w\mathbf{e}_z = U^{-1}(u_x\mathbf{e}_x + u_y\mathbf{e}_y + u_z\mathbf{e}_z)$, so that $\left\langle u^2+v^2+w^2\right\rangle = 1$, and dimensionless spatial coordinates $\ell^{-1}(x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z)$, and using the potential Φ and derivative multiplier Ψ we have the identity

$$\beta = \frac{\langle \Phi' u\rangle \left\langle \Psi uv + \frac{1}{Re}\Psi' u\right\rangle}{\langle \Phi\Psi\rangle}. \tag{3.5}$$

The upper bound β_b on the dissipation factor is obtained by first maximizing the right hand side of (3.5) over all normalized, divergence-free vector fields satisfying the boundary conditions, and then minimizing over all multiplier functions $\Psi \in H^1[0,1]$ satisfying homogeneous Dirichlet boundary conditions. Thus for any solution of the Navier-Stokes equations, $\beta \le \beta_b$ where the variational bound β_b is the solution of the mini-max problem

$$\beta_b(Re) = \min_{\Psi}\max_{\mathbf{u}} \frac{\langle \Phi' u\rangle \left\langle \Psi uv + \frac{1}{Re}\Psi' u\right\rangle}{\langle \Phi\Psi\rangle}. \tag{3.6}$$

Note that while we explicitly display the Reynolds number dependence of β_b, it also depends the shape of the applied force—but *not* independently on the forcing amplitude F; the ratios in (3.4), (3.5) and (3.6) are homogeneous in both ϕ and Φ.

3.2. *Evaluting bounds*

From (3.6) it follows immediately that $\beta_b(Re)$ is bounded by a function of the form $c_1 + c_2/Re$ for all Reynolds numbers, the analog of the result in Doering & Foias (2002). To see this, choose any convenient smooth multiplier function Ψ (e.g., Φ and observe that elementary Cauchy-Schwarz and Hölder estimates (recalling the unit normalization of $\mathbf{u}$) give

$$\langle \Phi' u\rangle \le \left\langle \phi^2\right\rangle^{1/2}, \quad \langle \Psi uv\rangle \le \frac{1}{2}\sup_{y\in[0,1]}|\Psi(y)|, \quad \langle \Psi' u\rangle \le \left\langle \Psi'^2\right\rangle^{1/2} \tag{3.7}$$

so that

$$\beta_b(Re) = \frac{\left\langle \phi^2\right\rangle^{1/2} \sup_{y\in[0,1]}|\Psi(y)|}{2\langle \Phi\Psi\rangle} + \frac{\left\langle \phi^2\right\rangle^{1/2}\left\langle \Psi'^2\right\rangle^{1/2}}{\langle \Phi\Psi\rangle} Re^{-1}. \tag{3.8}$$

This simple analysis produces explicit expressions for the coefficients c_1 and c_2 in a bound of the form $\beta \le c_1/Re + c_2$, displaying their functional dependence on the shape of the

driving force. In the following we will quantitatively and qualitatively improve this upper bound, computing the *exact* solution of the infinite Re limit of the mini-max problem.

To further estimate and evaluate $\beta_b(Re)$, note first that the boundary conditions together with incompressibility imply that the y-component satisfies $\overline{v}(y) \equiv 0$ where the overbar means horizontal and time average. We decompose the x-component into a horizontal mean flow $\overline{u}(y)$ and a fluctuating remainder $\tilde{u} = u - \overline{u}$. Then the terms in the numerator of the ratio for β_b reduce:

$$\langle \phi u \rangle = \langle \phi \overline{u} \rangle \, , \qquad \langle \Psi u v \rangle = \langle \Psi \tilde{u} v \rangle \, , \qquad \langle \Psi' u \rangle = \langle \Psi' \overline{u} \rangle \, . \tag{3.9}$$

Let $\xi^2 = \langle \overline{u}^2 \rangle$. The normalization for the velocity field is

$$1 = \left\langle \overline{u}^2 + \tilde{u}^2 + v^2 + w^2 \right\rangle \geq \xi^2 + \left\langle \tilde{u}^2 + v^2 \right\rangle , \tag{3.10}$$

so the terms in (3.9) may be estimated

$$|\langle \phi u \rangle| \leq \left\langle \phi^2 \right\rangle^{1/2} \xi, \qquad |\langle \Psi u v \rangle| \leq \frac{1}{2} \sup_{y \in [0,1]} |\Psi(y)| \, (1 - \xi^2), \qquad |\langle \Psi' u \rangle| \leq \left\langle \Psi'^2 \right\rangle^{1/2} \xi. \tag{3.11}$$

Hence for any choice of Ψ,

$$\max_{\mathbf{u}} \frac{\langle \Phi' u \rangle \left\langle \Psi u v + \frac{1}{Re} \Psi' u \right\rangle}{\langle \Phi \Psi \rangle} \leq \max_{0 \leq \xi \leq 1} \frac{\left\langle \phi^2 \right\rangle^{1/2}}{\langle \Phi \Psi \rangle} \xi \left[\frac{1}{2} \sup_{y \in [0,1]} |\Psi(y)| \, (1 - \xi^2) + \frac{1}{Re} \left\langle \Psi'^2 \right\rangle^{1/2} \xi \right] . \tag{3.12}$$

It is easy to find ξ_m, the maximizing value of ξ. It is the solution of a quadratic equation in the interval $[0, 1]$ for sufficiently high values of Re, or else it is $\xi_m = 1$ if

$$Re \leq 2 \frac{\left\langle \Psi'^2 \right\rangle^{1/2}}{\sup_{y \in [0,1]} |\Psi(y)|} . \tag{3.13}$$

When $\xi_m = 1$, the maximizing velocity field is a steady plane parallel flow, namely the Stokes flow for the given applied force. The right hand side of (3.13) is ≥ 4, providing, by a somewhat round-about derivation, a lower bound for the smallest possible critical Reynolds number of absolute stability of the steady plane parallel flow that is *uniform* in the shape of the applied shearing force. For the purposes of the discussion here, however, we use the estimates above to bound β_b as

$$\begin{aligned} \beta_b &\leq \min_{\Psi} \frac{\left\langle \phi^2 \right\rangle^{1/2}}{\langle \Phi \Psi \rangle} \left[\max_{0 \leq \xi \leq 1} \xi (1 - \xi^2) \frac{1}{2} \sup_{y \in [0,1]} |\Psi(y)| + \max_{0 \leq \xi \leq 1} \xi^2 \frac{1}{Re} \left\langle \Psi'^2 \right\rangle^{1/2} \right] \\ &= \min_{\Psi} \frac{\left\langle \phi^2 \right\rangle^{1/2}}{\langle \Phi \Psi \rangle} \left[\frac{1}{\sqrt{27}} \sup_{y \in [0,1]} |\Psi(y)| + \frac{1}{Re} \left\langle \Psi'^2 \right\rangle^{1/2} \right] . \end{aligned} \tag{3.14}$$

This leads to improved estimates, in terms of variational problems for an optimal multiplier Ψ and for c_2 in a bound of the form $\beta_b \leq c_1/Re + c_2$. As we will now show, the variational expression above for c_2 is sharp at high Reynolds numbers. That is, the $Re \to \infty$ limit of the extremization problem for the optimal Ψ and β_b can be solved exactly.

Define

$$\beta_b(\infty) = \min_{\Psi} \max_{\mathbf{u}} \frac{\langle \Phi' u \rangle \langle \Psi u v \rangle}{\langle \Phi \Psi \rangle} . \tag{3.15}$$

First we will evaluate $\beta_b(\infty)$, and then we will prove that

$$\limsup_{Re \to \infty} \beta_b(Re) \leq \beta_b(\infty). \tag{3.16}$$

We accomplish this through a series of two lemmas and a theorem.

Lemma 1: If $\phi = -\Phi' \in L^2[0,1]$ and $\Psi \in H^1[0,1]$ satisfies $\Psi(0) = 0 = \Psi(1)$, then

$$\max_{\mathbf{u}} \langle \Phi' u \rangle \langle \Psi uv \rangle = \frac{1}{\sqrt{27}} \left\langle \phi^2 \right\rangle^{1/2} \sup_{y\in[0,1]} |\Psi(y)| \tag{3.17}$$

where the velocity fields $\mathbf{u}$ are divergence-free and unit normalized in L^2, periodic in downstream (x) and spanwise (z) directions and free-slip in the normal (y) direction.

Proof: We may take $\Psi \neq 0$. The calculation for (3.14) already established that the proposed answer is an upper bound to this variational problem, so all we must do is display a sequence of acceptable test fields $\mathbf{u}$ that approach the bound. Note that any nonvanishing $\Psi \in H^1[0,1]$ satisfying the homogeneous Dirichlet conditions is uniformly continuous and its extremum is realized at a (not necessarily unique) point y_m in the open interval $(0,1)$. Consider the unit-normalized divergence-free vector field $\mathbf{u}_k$ with components

$$\begin{aligned} u_k &= g_k(y)\sqrt{2}\sin kz + \frac{1}{\sqrt{3}}\frac{\phi(y)}{\sqrt{\langle \phi^2\rangle}} \\ v_k &= \pm g_k(y)\sqrt{2}\sin kz \\ w_k &= \frac{1}{k} g_k'(y)\sqrt{2}\cos kz \end{aligned} \tag{3.18}$$

where $g_k(y)^2$ is a smooth approximation of a δ-function with compact support centered on y_m and normalized according to

$$\frac{1}{3} = \left\langle g_k^2 + \frac{g_k'^2}{2k^2} \right\rangle . \tag{3.19}$$

The wavenumber $k \neq 0$ is adjustable, and for each value of k we have $\langle \phi u_k \rangle = \frac{1}{\sqrt{3}} \left\langle \phi^2 \right\rangle^{1/2}$. Now $\langle \Psi u_k v_k \rangle = \langle \Psi \overline{u_k v_k} \rangle$ and we may concentrate $\overline{u_k v_k}(y) = \pm g_k(y)^2$ as tightly as desired around y_m by taking k large, in which case $\langle \Psi u_k v_k \rangle \to |\Psi(y_m)| \left\langle g_k^2 \right\rangle$. Moreover, $\left\langle g_k^2 \right\rangle \to 1/3$ as $k \to \infty$. Hence there exists a sequence of test fields $\mathbf{u}_k$ for which $\langle \phi u_k \rangle \langle \Psi u_k v_k \rangle$ approaches the upper bound. *QED.*

Lemma 2: If $\Phi \in H^1[0,1]$ and sign $[\Phi(y)]$ has a finite number of discontinuities, then

$$\min_{\Psi\in H^1[0,1]} \frac{\sup_{y\in[0,1]} |\Psi(y)|}{\langle \Phi\Psi \rangle} = \frac{1}{\langle |\Phi| \rangle}. \tag{3.20}$$

The minimizing function is $\Psi_m(y) = \mathrm{sign}\,[\Phi(y)]$ which is not in $H^1[0,1]$, but it is the pointwise limit of a sequence of functions in H^1.

Proof: Note that

$$\langle \Phi\Psi \rangle \leq \sup_{y\in[0,1]} |\Psi(y)| \, \langle |\Phi| \rangle \tag{3.21}$$

so the proposed answer is a lower bound to the minimum, and the function $\Psi_m(y) = \mathrm{sign}\,[\Phi(y)]$ saturates this bound. Then it is straightforward to mollify the finite number of discontinuties in Ψ_m to produce a sequence of H^1 functions converging pointwise to Ψ_m. Let the number of discontinuities of $\Psi_m(y)$ be N, located in order at y_n. We may smooth Ψ_m by introducing a finite slope near each y_n to produce the regulated function $\Psi_\delta(y)$ sketched in figure 1. The mollified $\Psi_\delta(y)$ is linear with slope $\pm\delta^{-1}$ inside all the intervals 2δ around each y_n and within δ of the ends of the interval at $y = 0$ and 1.

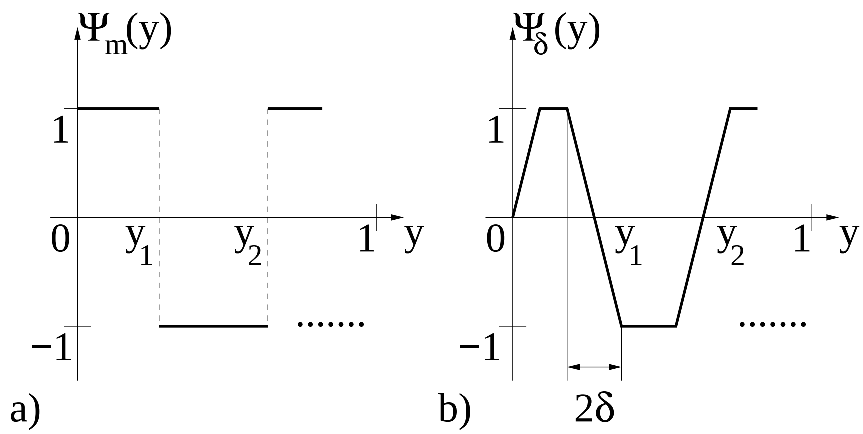

FIGURE 1. Boundary layer regularization for the optimal multiplier Ψ_m. (a) Sketch of $\Psi_m(y)$ with a finite number of jumps in [0,1] at $y = 0, y_1, y_2, \dots, 1$. (b) Sketch of the mollified $\Psi_\delta(y)$ that converges pointwise to $\Psi_m(y)$ for $\delta \to 0$.

Because $\Phi \in H^1[0,1]$, $\Phi = \mathcal{O}(\sqrt{|y-y_n|})$ near the isolated zeros y_n where Ψ_m jumps. Thus $\langle \Phi\Psi_\delta\rangle = \langle|\Phi|\rangle\left[1-\mathcal{O}(\delta^{3/2})\right]$. Although $\left\langle(\Psi_\delta')^2\right\rangle = \mathcal{O}(\delta^{-1})$, for each $y \in [0,1]$, $\Psi_\delta(y) \to \Psi_m(y)$ as $\delta \to 0$. *QED.*

Theorem: If $\Phi \in H^1[0,1]$ such that sign$\,[\Phi(y)]$ has a finite number of jump discontinuities, then

$$\beta_b(\infty) \;=\; \frac{1}{\sqrt{27}}\;\frac{\sqrt{\langle\phi^2\rangle}}{\langle|\Phi|\rangle} \tag{3.22}$$

and

$$\beta_b(Re) \;\le\; \beta_b(\infty) \;+\; \mathcal{O}(Re^{-3/4}) \tag{3.23}$$

as $Re \to \infty$.

Proof: Lemmas 1 and 2 establish the value of $\beta_b(\infty)$ in (3.22). To establish (3.23), recall from the proof of Lemma 2 that $\langle\Phi\Psi_\delta\rangle = \langle|\Phi|\rangle\left[1-\mathcal{O}(\delta^{3/2})\right]$ and $\left\langle(\Psi_\delta')^2\right\rangle = \mathcal{O}(\delta^{-1})$. Using these facts together with Lemma 1,

$$\begin{aligned}
\beta_b(Re) &= \min_{\Psi}\max_{\mathbf{u}} \frac{\langle\Phi' u\rangle\left\langle\Psi uv + \frac{1}{Re}\Psi' u\right\rangle}{\langle\Phi\Psi\rangle}\\
&\le \min_{\delta}\max_{\mathbf{u}} \frac{\langle\Phi' u\rangle\left\langle\Psi_\delta uv + \frac{1}{Re}\Psi'_\delta u\right\rangle}{\langle\Phi\Psi_\delta\rangle}\\
&\le \min_{\delta}\max_{\mathbf{u}} \frac{\langle\Phi' u\rangle\left(\langle\Psi_\delta uv\rangle + \frac{1}{Re}\mathcal{O}(\delta^{-1/2})\right)}{\langle\Phi\Psi_\delta\rangle}\\
&\le \min_{\delta}\left(\frac{1}{\sqrt{27}}\frac{\left\langle\phi^2\right\rangle^{1/2}}{\langle|\Phi|\rangle}\left[1+\mathcal{O}(\delta^{3/2})\right]\left[1+\frac{1}{Re}\mathcal{O}(\delta^{-1/2})\right]\right).
\end{aligned} \tag{3.24}$$

Choosing $\delta = \mathcal{O}(Re^{-1/2})$ establishes the result. *QED.*

We make three short technical remarks here:

(i) Although we only showed that $\limsup_{Re\to\infty}\beta_b(Re) \le \beta_b(\infty)$, it is natural to conjecture that at finite Re the optimal multiplier Ψ actually looks like the mollified multipliers Ψ_δ and that $\lim_{Re\to\infty}\beta_b(Re) = \beta_b(\infty)$. But this remains to be proven.

(ii) The $\mathcal{O}(Re^{-3/4})$ rate of approach to the $Re \to \infty$ limit in the theorem is not optimal for smoother shape functions. This is easy to see by repeating the proof of the theorem assuming, say, that $\phi \in H^1$ so that Φ has a bounded derivative and behaves linearly (rather than as a square root) near its zeros. That generic linear behavior leads to a faster $\mathcal{O}(Re^{-4/5})$ rate.

(iii) The hypothesis of a finite number of zeros in Φ is probably not really necessary given $\Phi \in H^1$; we invoke it here for simplicity of the proofs only. In any case, for the applications we have in mind, ϕ and Φ will actually be extremely smooth (composed, for example, of a finite number of Fourier components) so the theorem as stated and proved here serves our purposes.

4. Comparison with numerical results and discussion

Direct numerical simulations (DNS) in this geometry with these kinds of forces are possible in Fourier space thanks to the free-slip boundary conditions. For computations we used the pseudospectral code developed in Schumacher & Eckhardt (2000) and Schumacher & Eckhardt (2001) with numerical resolution of $256 \times 65 \times 256$ grid points. The steady volume forcing density, $\mathbf{f}(\mathbf{x})$, was chosen such that a laminar (and linearly stable!) shear flow profile $\mathbf{u}_0(\mathbf{x}) = -U_0 \cos(\pi y/\ell)\mathbf{e}_x$ could be sustained. From the Navier-Stokes equations (2.1) it follows for this plane-parallel shear flow that

$$\mathbf{f}(\mathbf{x}) = -\frac{\nu U_0 \pi^2}{\ell^2} \cos(\pi y/\ell)\mathbf{e}_x = F\phi\left(\frac{y}{\ell}\right)\mathbf{e}_x \tag{4.1}$$

with shape $\phi(\eta) = \sqrt{2}\cos\pi\eta$ and amplitude $F = -\nu U_0 \pi^2/\sqrt{2}\ell^2$. The flow can be considered a Kolmogorov flow (see Borue & Orszag (1996) and Childress, Kerswell & Gilbert (2001)) with additional symmetry constraints in the normal (y) direction. The aspect ratio and scales for the calculations were $L_x/\ell = L_z/\ell = 2\pi$. The Grashof number for absolute (energy) stability of the steady plane-parallel flow with this force shape is $Gr_c = 68$ where the Reynolds number is less than 7; the simulations were carried out well above this value, for Gr varying between 4900 and 59200.

Mathematical results for the shape function $\phi(y) = -\sqrt{2}\cos\pi y$ (equivalently $\Phi(y) = \pi^{-1}\sqrt{2}\sin\pi y$) are shown in Figure 2 along with the DNS results. In contrast to shear flows driven by rigid walls where the friction (dissipation) factor tends to decrease with increasing Reynolds number, here we observe a slight increase. The numerical values for β are about a factor 3 below the upper bound. This is a significantly better comparison of the data and the bounds than for turbulent Couette flow where the discrepancy is a factor 10 at $Re \approx 10^6$.

The mean profiles of the streamwise velocity component for two different Reynolds numbers are shown in the inset in Figure 2. It is interesting to note that even though the force shape is nonlinear across the layer, the mean profile is relatively linear with mean shear nearly constant outside boundary layers near the no-slip walls. The high Reynolds number limit of the optimal multiplier $\psi_m(y)$ is piecewise linear with constant magnitude of its slope function; $|\psi'_m(y)| = |\Psi_m(y)| = 1$ away from the corners. We point out the similarity here with the observed mean profiles for the single example we have at hand.

It will be very interesting to study the bounds $\beta_b(Re)$ as well as the optimal multiplier functions at finite Reynolds numbers for a variety of force shape functions ϕ. This is because while the behavior of the bound on β is similar in structure to the observed experimental and computational values (Sreenivasan (1984, 1998)), it remains an open question how the high-Re value of β depends on the details of the driving. There are some features we can anticipate right away, though. Assuming that the structure of

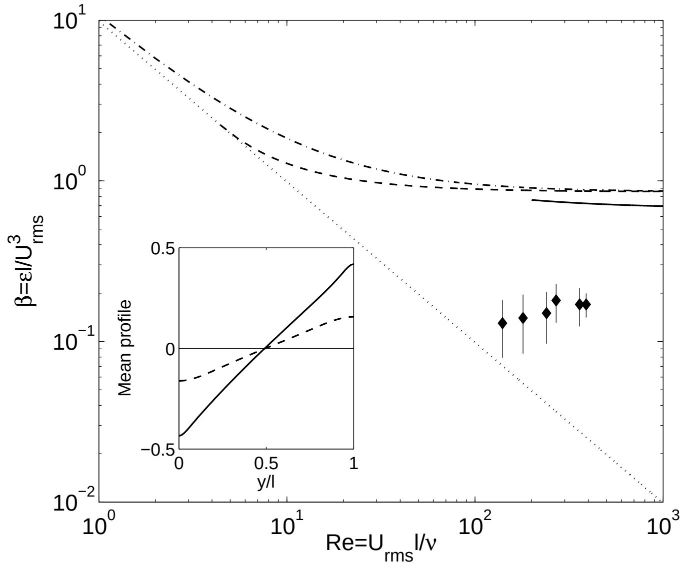

FIGURE 2. The dissipation factor $\beta = \epsilon\ell/U^3$ as a function of the Reynolds number $Re = U\ell/\nu$ for the force shape function $\phi(y/\ell) = -\sqrt{2}\cos\pi y/\ell$. The results of the direct numerical simulations are indicated by diamonds with error bars due to standard deviation $\pm\sigma_\beta$ where $\sigma_\beta = \beta(\sigma_\epsilon/\epsilon + 3\sigma_{U_{rms}}/U_{rms})$. The lower dotted line is the dissipation in the steady laminar flow which is a lower limit to the dissipation factor for any (statistically) stationary flow with this force. The three upper bounds, from top to bottom, are the estimate in (3.14) with the choice $\Psi = \Phi$ (dash-dot line), the estimate in (3.12) with the exact maximization over ξ followed by the choice $\Psi = \Phi$ (dashed line), and the optimal value $\beta_b(\infty) = \pi^2/\sqrt{216}$ from the theorem with the rigorous $\mathcal{O}(Re^{-4/5})$ approach added on (solid). The optimal bound for the infinite Re limit is a 22% improvement below the infinite Re limit of the bound with $\Psi = \Phi$. The mean flow profiles $\overline{u_x}(y)$ for the simulations with the smallest (dashed) and largest (solid) Re are shown in the inset.

$\Psi_m(y)$ persists for large but finite Reynolds numbers, the high-Re optimal multiplier is a simple but interesting nonlinear functional of the shape function of the driving force. While the plane-parallel Stokes flow profile $U_{Stokes}(y)$ is a linear functional of the shape function,

$$U_{Stokes}(y) \sim \int_0^y dy' \left[\int_0^{y'} dy''\phi(y'')\right] + C, \tag{4.2}$$

the (infinite Re) optimal multiplier comes from a curiously similar—but highly nonlinear—formula:

$$\psi_m(y) \sim \int_0^y dy' \mathrm{sign}\left[\int_0^{y'} dy''\phi(y'')\right] + C. \tag{4.3}$$

This expression for ψ_m displays bifurcations as a functional of the shape function ϕ. That is, for some shape functions, variations in ϕ may result in no change at all in the associated high-Re optimal multiplier, while at other configurations small changes in ϕ can produce large changes in ψ_m (such as the number of “kinks” in the multiplier profile). Whether or not this kind of effect reflects any features of high Reynolds number mean profiles for shear turbulence driven by other shaped forces remains to be seen.

To summarize, in this paper we have derived and analyzed a variational mini-max problem for upper bounds on the energy dissipation rate valid for both low and high

Reynolds number (including turbulent) body-forced shear flows. We find that the maximizing flow fields are characterized by streamwise vorticies concentrated near the maximal shear in an auxillary "multiplier" profile, analogous to the "background" profile utilized in Doering & Constantin (1994), Nicodemus *et al* (1998), Kerswell (1998) and Childress, Kerswell & Gilbert (2001). We solved the optimal high Re mini-max problem exactly and compared the results with data from direct numerical simulations for a specific choice of forcing. We observed that the high Re bound is only about a factor of three above the data, and also that the high Re optimal multipier shares some qualitative features with the measured mean flow profiles. Future work in this area will include investigations for other force shapes, as well as the improvement of rigorous bounds by exact numerical evaluation and/or by the inclusion of a balance parameter (see Nicodemus *et al* (1997)) in the variational problem. Finally, we remark that although we have carried out the bounding analysis with a steady driving for the flow, there is no obstruction to the inclusion of time-dependent forcing in the model.

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